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TABLES FOR COMPUTING VARIOUS CASES OF BEAM COLUMNS

By J. Cassens

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TABLES FOR COMPUTING VARIOUS CASES OF BEAM COLUMNS*

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For a better understanding of these tables, the methods by which the cases are computed are discussed first. The importance of the buckling modulus is pointed out.

I. EXPLANATION OF THE METHODS

Aircraft design methods differ from other structural engineering methods in the selection of slender beams. Since the former came into being at a time when structural engineering had already reached certain standards, it is not surprising that the stress analysis in aircraft design was largely influenced by the other. But it is surprising that Müller-Breslau, for instance, annexed the column tests in his work "Graphical Statics of Building Construction" on the subject of aircraft design. The slender spars of braced wings, the slender members of steel-tube bodies and of the ribs no longer permit the calculation of members in rough approximation first for bending and then for buckling, but the analysis had to correspond to actual loading conditions, that is, combined bending and column stress. The importance attached to this subject in past years is readily seen from a glance at the old Z.F.M. (Zeitschrift für Flugtechnik und Motorluftschiffahrt). For example, in 1918 and 1919 these problems were treated by Gümbel, Pröll, Trefftz, Müller-Breslau, Reissner, Ratzersdorfer, Arnstein, Koenig, etc. But the authors really dealt merely with the problem shown in the present report below no. 4 with corresponding applications.

It is true that in its principal beams modern aircraft design again tends toward short columns, but even so, column-effect problems remain to be solved.

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*"Tafel einiger Knickbiegefälle." Luftfahrtforschung, vol. 18, nos. 2 and 3, March 29, 1941, pp. 86-94.

Column effect and its counterpart, bending tension, deals with the case of concurrent longitudinal and transverse forces acting on a member. Hütte and various other manuals contain tables which give the transverse forces, moments, deflections, and various other data for bending forces on the straight member with constant bending stiffness. These forces are supplemented by the effect of the moment Py, where P denotes the longitudinal force and y the deflection at any one point. The relation between curvature radius ρ and bending forces reads:

$$+ \frac{1}{\rho} = - \frac{d^2y}{dx^2} = - \frac{M_x}{EJ}$$

where M_x is the moment of all bending forces dependent upon beam ordinate x, EJ the bending stiffness, and y the deflection of the neutral axis. (See fig. 1.) Dividing moment M_x into a portion $(\underline{M}_x)$ due solely to the transverse forces and a portion affected only by P, affords $M_x = \underline{M}_x + Py$, which, posted in the foregoing relation, gives

$$k^2 \; y'' + y = - \frac{\underline{M}_x}{P} \tag{1}$$

with $k^2 = \frac{EJ}{P}$. (The investigations apply to constant bending stiffness EJ and constant longitudinal force P.) The simplest way of solving the differential equation is as follows: The homogeneous differential equation is expressed as $y = C\ e^{ax}$; the values C are coefficients to be defined later and must satisfy the boundary conditions of the problem, while a establishes the connection between the formula and the original equation (1). The complete differential equation is solved either by power formulas, the coefficients of which are determined by comparison, or else by others, for instance, trigonometrical ones, depending upon the character of the right side of equation (1). The majority of problems shown in the tables were solved by this method. The right side of equation (1) shows that the law of superposition holds for all loads applied transverse to the neutral axis. Each loading condition is to be computed for applied longitudinal force P.

The law of superposition can be made clever use of in the calculation of a beam with constant bending stiffness and longitudinal force, the beam being transversely loaded by several, arbitrarily many, concentrated loads. This merely requires expressions for deflections and moments of a beam transversely loaded by one concentrated load, as found under no. 13.

Under a concentrated load Q_0 the moment and the deflection to the left of the concentrated load is

$$M_x = Q_0 \, k \, \frac{\sin \eta}{\sin \alpha} \sin \varphi$$

$$y = \frac{Q_0}{P} \left(k \, \frac{\sin \eta}{\sin \alpha} \sin \varphi - \frac{d}{l} \, x \right)$$

and to the right

$$M_u = Q_0 \, k \, \frac{\sin \zeta}{\sin \alpha} \sin \psi$$

$$v = \frac{Q_0}{P} \left(k \, \frac{\sin \zeta}{\sin \alpha} \sin \psi - \frac{b}{l} \, u \right)$$

Therefore, if a beam is under transverse load Q_3, Q_4, and Q_5 in addition to the longitudinal force P (fig. 2), the moment at point $x_0 \left(\varphi_0 = \frac{x_0}{k} \right)$ is:

$$M_{x_0} = \frac{k \sin \varphi_0}{\sin \alpha} \left(Q_3 \sin \eta_3 + Q_4 \sin \eta_4 + Q_5 \sin \eta_5 \right)$$

and the deflection

$$y_0 = \frac{k \sin \varphi_0}{P \sin \alpha} \left(Q_3 \sin \eta_3 + Q_4 \sin \eta_4 + Q_5 \sin \eta_5 \right) - \frac{x_0}{P l} \left(Q_3 d_3 + Q_4 d_4 + Q_5 d_5 \right)$$

As a result of transverse loads Q_1 and Q_2 with P (fig 2), the moment at point $u_0 = l - x_0 \left(\psi_0 = \frac{u_0}{k} \right)$ is

$$M_{u_o} = \frac{k \sin \psi_o}{\sin \alpha} (Q_1 \sin \zeta_1 + Q_2 \sin \zeta_2)$$

and the deflection

$$v_o = \frac{k \sin \psi_o}{P \sin \alpha} (Q_1 \sin \zeta_1 + Q_2 \sin \zeta_2) - \frac{u_o}{P l} (Q_1 b_1 + Q_2 b_2)$$

If Q_1, Q_2, Q_3, Q_4, and Q_5 are applied simultaneously, the moment at $x_o = l - u_o$ is

$$M_{x_o, u_o} = M_{x_o} + M_{u_o}$$

and the deflection δ at this point $x_o = l - u_o$ is:

$$\delta = y_o + v_o$$

This problem is naturally much more difficult if longitudinal forces are applied at the same points as the transverse forces. In this case the use of an expanded Clapeyron equation such as cited by Müller-Breslau in his "Graphical Statics," vol. 2, 2, p. 643, is recommended. Examples for such problems may be found in the Z.F.M., 1920, p. 283.

A second method for column stress calculation is given because the final results appear in substantially different form. Instead of trigonometric and hyperbolic functions, a quotient appears which gives the effect of P in the denominator. Bleich, for example, computes the central moment of a beam loaded, as in case no. 3, according to formula

$$M_{max} = \frac{g l^2}{8} \left(1 + \frac{1.032}{P_E/P - 1}\right) = \frac{g l^2}{8} \frac{P_E/P + 0.032}{P_E/P - 1}$$

But no advantage accrues from this method except for symmetric load cases.

A third method is based upon the application of the "passive energy" or the principle of virtual speed (Föppl, "Drang and Zwang," vol. I, p. 61). At first sight the idea seems attractive, since the energy of the longitudinal forces × the deflection can be discounted in first approximation. But the application of the method involves a great number of difficulties.

According to this principle the deflection of a beam follows from the relation

$$1\ \delta_{ik} = \int \frac{M_i M_k}{E\ J}\, dx$$

The method is best explained by the example illustrated in figure 3.

The work performed in the beam is computed after the actual load Q and P is applied on the beam deflected under load 1. It affords

$$1\ \delta = \int_0^l \frac{(Q\ x + P\ y)\ (1\ x)}{E\ J}\, dx$$

or

$$\delta = \int_0^l \frac{(Q\ x)\ (x)}{E\ J}\, dx + \int_0^l \frac{(P\ y)\ (x)}{E\ J}\, dx$$

The first integral is the same as that obtained from computing the beam deflection under transverse load Q; let us call it δ_o. The second integral is a function of the looked-for deflection δ.

Herewith the relation can be transformed into

$$\delta \left(1 - \int_0^l \frac{P\ \frac{y}{\delta}\ x}{E\ J}\, dx\right) = \delta_o \tag{2}$$

The expression

$$\int_0^l \frac{P \frac{y}{\delta} x}{E J} dx$$

can be found by trial; y is expressed in a formula

$$\frac{y}{\delta} = \sin\left(\frac{\pi}{2} \frac{x}{l}\right)$$

which satisfies among others the boundary conditions for deflection at $x = 0$ and $x = l$ as well as for the tangent at $x = l$.

For constant P and EJ it affords

$$\frac{P}{EJ} \int_0^l x \sin\left(\frac{\pi}{2} \frac{x}{l}\right) dx = \frac{P}{EJ} \frac{4 l^2}{\pi^2} = 4 \frac{P}{P_E}; \quad P_E = \frac{E J \pi^2}{l^2}$$

The result is a very close approximation; hence

$$\delta \left(1 - 4 \frac{P}{P_E}\right) = \delta_0; \quad \delta = \frac{\delta_0}{1 - 4 \frac{P}{P_E}}$$

Very small values δ_0 are accompanied by an appreciable deflection δ if the denominator expression is likewise very small. If the denominator becomes equal to zero, it affords an expression for the stability condition of this problem, which is generally known.

It gives

$$P_k = \frac{P_E}{4} = \frac{1}{4} \frac{E J \pi^2}{l^2}$$

Tension bending can also be very readily computed by this method. With P indicating a tension, equation (2) reads:

$$\delta \left(1 + \int_0^l \frac{P \frac{y}{\delta} x}{E J} dx\right) = \delta_o \qquad (2a)$$

Whether a column effect case according to this method, is easy or difficult to compute depends upon the success with the formula for $\frac{y}{\delta}$. The Fourier analysis can be used by analyzing the expression for the deflection without the effect of P according to Fourier.

If the central moment of a beam loaded, as in case no. 3, is computed by this method, it affords first for the deflection

$$\delta_m = \frac{5}{384} \frac{g l^4}{E J} \frac{P_E/P}{P_E/P - 1}$$

and for the moment

$$M_{max} = \frac{g l^2}{8} \frac{P_E/P + 0.0281}{P_E/P - 1}$$

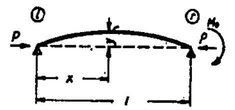
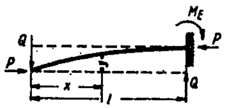
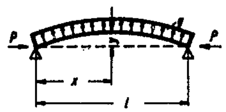
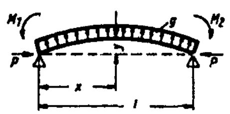
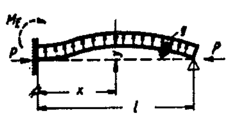
which is in good agreement with Bleich's result.

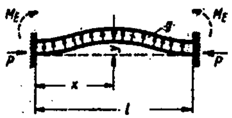
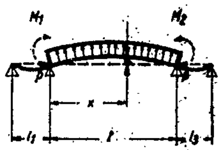
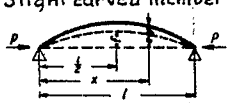
II. MODULUS OF BUCKLING UNDER COLUMN EFFECT

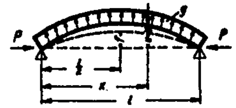
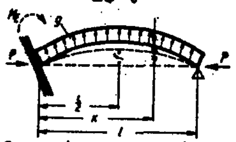
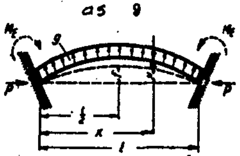
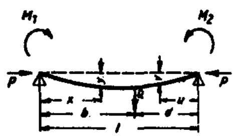
Occasionally it happens that comparatively short columns must be analyzed for column effect where $\sigma = \frac{P}{F}$ is higher than the break between Tetmayer's straight line and Euler's hyperbola in the $\sigma_k = f(\lambda)$ diagram. In such cases it has been found practical to replace E modulus by the buckling modulus. (See the writer's article "Column Testing," Luftfahrtforschung, vol. 17, no. 10, 1940, pp. 306-313.) To retain the E-modulus in such a case would afford much too great a safety relative to the column load according to Euler's hyperbola, while from pure buckling tests it is known that the member can take up only one load according to the Tetmayer straight line.

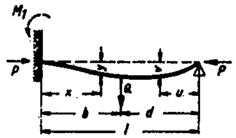
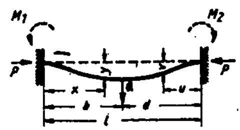
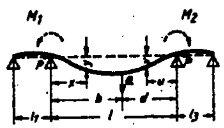
Under comparatively low longitudinal stresses $\frac{P}{F}$ the E-modulus can be used even if certain parts of the section due to combined bending and longitudinal stress assume values within Tetmayer's range. In such cases the effect is the opposite to that of pure crippling of short columns, that is, the bending factor becomes effective. The bending factor takes into consideration the experience with bending tests where it was found that the measured M_{B_r} was greater than computed according to the expression $\sigma_B W$, where σ_B is the material strength from tensile tests.

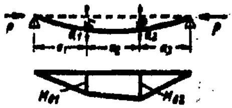
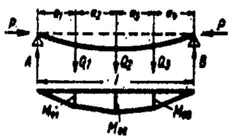
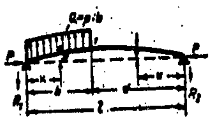
Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

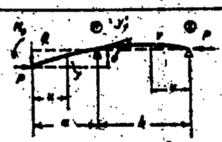
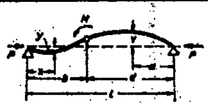
Nr.	Problem graph identification	Deflection y (P time value) Tangent $\frac{dy}{dx}$ ($P\cdot k$ " ") 2. Derivation $\frac{d^2y}{dx^2}$ (Pk^2 " ")	Moments M_x; $M_x = -E\cdot J\frac{d^2\cdot y}{dx^2}$; Special Moments	Constants A and B Member buckles under
1	 $\varphi = \frac{x}{k};\ \alpha = \frac{l}{k};\ k = \sqrt{\frac{EJ}{P}}$	**Compression bending** $P\cdot y = B\cdot\sin\varphi - M_0\frac{x}{l}$ $Pky' = B\cos\varphi - M_0\frac{k}{l}$ $Pk^2\cdot y'' = -B\cdot\sin\varphi$ $y'_l = \frac{M_0}{P\cdot l}\left(\frac{\alpha}{\sin\alpha}-1\right)$ $y'_r = -\frac{M_0}{P\cdot l}\left(1-\frac{\alpha}{\tan\alpha}\right)$ **Tension bending** $P\cdot y = B\cdot\mathfrak{Sin}\,\varphi + M_0\frac{x}{l}$ $P\cdot ky' = B\cdot\mathfrak{Cos}\,\varphi + M_0\frac{k}{l}$ $Pk^2y'' = +B\cdot\mathfrak{Sin}\,\varphi$ $y'_l = +\frac{M_0}{P\cdot l}\left(1-\frac{\alpha}{\mathfrak{Sin}\,\alpha}\right)$ $y'_r = -\frac{M_0}{P\cdot l}\left(\frac{\alpha}{\mathfrak{Tg}\,\alpha}-1\right)$	**Compression bending** $M_x = M_0\cdot\frac{x}{l} + P\cdot y$ $M_x = B\cdot\sin\varphi = \frac{M_0}{\sin\alpha}\cdot\sin\varphi$ $M_{max} = \frac{M_0}{\sin\alpha}$ (if $\alpha > 90^0$ ist) At point $x_0 = k\cdot\frac{\pi}{2}$ **Tension bending** $P =$ Tension $M_x = M_0\frac{x}{l} - P\cdot y$ $M_x = -B\cdot\mathfrak{Sin}\,\varphi = +\frac{M_0}{\mathfrak{Sin}\,\alpha}\cdot\mathfrak{Sin}\,\varphi$	**Compression** $B = \frac{M_0}{\sin\alpha}$ **Tension** $B = -\frac{M_0}{\mathfrak{Sin}\,\alpha}$ $\alpha = 180^0$
2		$P\cdot y = B\cdot\sin\varphi - Q\cdot x$ $P\cdot ky' = B\cdot\cos\varphi - Q\cdot k$ $Pk^2y'' = -B\sin\varphi$	$M_x = Q\cdot x + P\cdot y$ $M_x = B\cdot\sin\varphi = \frac{Q\cdot k}{\cos\alpha}\cdot\sin\varphi$ $M_{max} = M_E = Q\cdot k\cdot\mathrm{tg}\,\alpha$	$B = \frac{Q\,k}{\cos\alpha}$ $\alpha = 90^0$
3		$P\cdot y = A\cdot\cos\varphi + B\cdot\sin\varphi - \frac{g}{2}(l\,x - x^2) - g\,k^2$ $P\,k\,y' = -A\sin\varphi + B\cos\varphi - \frac{g\,k}{2}\,(l-2\,x)$ $P\,k^2\,y'' = -A\cos\varphi - B\sin\varphi + g\,k^2$	$M_x = \frac{g}{2}(l\cdot x - x^2) + P\cdot y$ $M_x = A\cos\varphi + B\cdot\sin\varphi - g\,k^2$ $M_{max} = g\,k^2\left(\frac{1}{\cos\frac{\alpha}{2}} - 1\right)$ at center	$A = +g\,k^2$ $B = +g\,k^2\cdot\mathrm{tg}\,\frac{\alpha}{2}$ $\alpha = 180^0$
4		$P\cdot y = A\cdot\cos\varphi + B\sin\varphi$ $- \frac{g}{2}(l\cdot x - x^2) - M_1 - (M_2 - M_1)\frac{x}{l} - g\,k^2$ $P\,k\,y' = -A\sin\varphi + B\cos\varphi$ $-\frac{g\,k}{2}(l-2\,x) - (M_2 - M_1)\frac{k}{l}$ $P\,k^2\,y'' = -A\cos\varphi - B\cdot\sin\varphi + g\,k^2$	$M_x = \frac{g}{2}(l\cdot x - x^2) + M_1 + (M_2 - M_1)\frac{x}{l} + P\cdot y$ $M_x = A\cdot\cos\varphi + B\cdot\sin\varphi - g\,k^2$ M_{max} bei tang $\frac{x}{k} = \frac{B}{A} = \mathrm{tang}\,\varphi_0$ at point $x_0 = k\cdot\varphi_0$ $M_{max} = \frac{A}{\cos\varphi_0} - g\,k^2 = g\,k^2\left(\frac{1}{\cos\varphi_0} - 1\right) + \frac{M_1}{\cos\varphi_0}$	$A = +g\,k^2 + M_1$ $B = \frac{g\,k^2 + M_2}{\sin\alpha} - \frac{g\,k^2 + M_1}{\mathrm{tg}\,\alpha}$; or $B = \frac{M_2 - M_1\cos\alpha}{\sin\alpha} + g\,k^2\cdot\mathrm{tg}\,\frac{\alpha}{2}$ $\alpha\ 180^0$
5	 Find : M_E	$P\cdot y = A\cdot\cos\varphi + B\cdot\sin\varphi$ $-\frac{g}{2}(l\cdot x - x^2) + M_E\left(1 - \frac{x}{l}\right) - g\,k^2$ $P\cdot k\cdot y' = -A\sin\varphi + B\cdot\cos\varphi - \frac{g\,k}{2}(l-2\,x) - M_l\cdot\frac{k}{l}$ $P\,k^2\,y'' = -A\cos\varphi - B\cdot\sin\varphi + g\,k^2$	$M_x = \frac{g}{2}(l\cdot x - x^2) - M_E\left(1 - \frac{x}{l}\right) + P\cdot y$ M_{max} bei $\mathrm{tg}\,\frac{x}{k} = \frac{B}{A} = \mathrm{tang}\,\varphi_0$ at point $M_x = A\cdot\cos\varphi + B\cdot\sin\varphi - g\,k^2$ $\quad x_0 = k\cdot\varphi_0$ $M_E\left(1 - \frac{\alpha}{\mathrm{tg}\,\alpha}\right) = g\,l^2\left(\frac{1-\cos\alpha}{\alpha\cdot\sin\alpha} - \frac{1}{2}\right)$ $M_{max} = \frac{A}{\cos\varphi_0} - g\,k^2 = g\,k^2\left(\frac{1}{\cos\varphi_0} - 1\right) - \frac{M_E}{\cos\varphi_0}$	$A = +g\,k^2 - M_E$ $B = +g\,k^2\,\mathrm{tg}\,\frac{\alpha}{2} + \frac{M_E}{\mathrm{tg}\,\alpha}$ $\alpha = 257^0\,30'$

Nr.	Problem graph identification	Deflection y (P time value) Tangent $\frac{dy}{dx}$ ($P \cdot k$ " ") 2. Derivation $\frac{d^2y}{dx^2}$ ($P k^2$ " ")	Moments $\begin{cases} M_x \\ M_x = -E \cdot J \frac{d^2 \cdot y}{dx^2} \end{cases}$ Special Moments	Constants A and B Member buckles under
6	 Find: M_E	$P \cdot y = A \cdot \cos\varphi + B \cdot \sin\varphi - \frac{g}{2}(l \cdot x - x^2) + M_E - g k^2$ $P k y' = -A \sin\varphi + B \cos\varphi - \frac{g k}{2}(l - 2x)$ $P \cdot k^2 y'' = -A \cos\varphi - B \cdot \sin\varphi + g k^2$	$M_x = \frac{g}{2}(l \cdot x - x^2) - M_E + P \cdot y$ $M_x = A \cos\varphi + B \cdot \sin\varphi - g k^2$ At center $M_M = g k^2 \cdot \left(\frac{\alpha}{2} \cdot \frac{1}{\sin\frac{\alpha}{2}} - 1\right)$ $M_E = g k^2 \left(1 - \frac{\alpha}{2} \cdot \frac{1}{\operatorname{tg}\frac{\alpha}{2}}\right) = g k^2 \left(1 - \frac{\alpha}{2} \frac{1 + \cos\alpha}{\sin\alpha}\right)$ or $M_E \left(\frac{\alpha}{\sin\alpha} - \frac{\alpha}{\operatorname{tg}\alpha}\right) = g l^2 \cdot \left(\frac{1 - \cos\alpha}{\alpha \cdot \sin\alpha} - \frac{1}{2}\right)$	$A = + g k^2 - M_E$ $B = (g k^2 - M_E) \cdot \operatorname{tg}\frac{\alpha}{2}$ $\alpha = 360^0$
7	 long. force only in field l. Find: M_1 and M_2 $y'_{x=0} = y'_1$; $y'_{x=l} = y'_2$ In field l_1: J_1; $E . J_1 = B_1$ In field l_3: J_3; $E . J_3 = B_3$ In central field: $K = \sqrt{\frac{EJ}{P}}$; $\varphi = \frac{x}{K}$; $\alpha = \frac{l}{K}$	$P \cdot y = A \cdot \cos\varphi + B \cdot \sin\varphi$ $- \frac{g}{2}(l \cdot x - x^2) + M_1 + (M_2 - M_1)\frac{x}{l} - g k^2$ $P \cdot k y' = -A \sin\varphi + B \cdot \cos\varphi$ $- \frac{g k}{2}(l - 2x) + (M_2 - M_1) \cdot \frac{k}{l}$ $P k^2 \cdot y'' = -A \cos\varphi - B \cdot \sin\varphi + g k^2$	$M_x = \frac{g}{2}(l \cdot x - x^2) - M_1 - (M_2 - M_1)\frac{x}{l} + P \cdot y$ $M_x = A \cdot \cos\varphi + B \cdot \sin\varphi - g k^2$ from $P \cdot k y_1'$ and $P \cdot k y_2'$ the equations for M_1 and M_2 read $M_1 \left(1 - \frac{\alpha}{\operatorname{tg}\alpha}\right) + M_2 \left(\frac{\alpha}{\sin\alpha} - 1\right) = g l^2 \left(\frac{1}{\alpha} \cdot \operatorname{tg}\frac{\alpha}{2} - \frac{1}{2}\right) - y_1' \cdot P \cdot l$ $M_1 \left(\frac{\alpha}{\sin\alpha} - 1\right) + M_2 \left(1 - \frac{\alpha}{\operatorname{tg}\alpha}\right) = g l^2 \left(\frac{1}{\alpha} \cdot \operatorname{tg}\frac{\alpha}{2} - \frac{1}{2}\right) - y_2' \cdot P \cdot l$ $y_1' = \frac{M_1 \cdot l_1}{3 B_1}$; $y_2' = \frac{M_2 \cdot l_3}{3 B_3}$ $M_1 \left[P \cdot l \cdot \frac{l_1}{3 B_1} + 1 - \frac{\alpha}{\operatorname{tg}\alpha}\right] + M_2 \left(\frac{\alpha}{\sin\alpha} - 1\right) = g l^2 \cdot \left(\frac{1}{\alpha} \cdot \operatorname{tg}\frac{\alpha}{2} - \frac{1}{2}\right)$ (1) $M_1 \left(\frac{\alpha}{\sin\alpha} - 1\right) + M_2 \cdot \left[P \cdot l \frac{l_3}{3 B_3} + 1 - \frac{\alpha}{\operatorname{tg}\alpha}\right] = g l^2 \cdot \left(\frac{1}{\alpha} \cdot \operatorname{tg}\frac{\alpha}{2} - \frac{1}{2}\right)$ (2)	$A = + g k^2 - M_1$ $B = + g k^2 \cdot \operatorname{tg}\frac{\alpha}{2} - \frac{M_2 - M_1 \cos\alpha}{\sin\alpha}$ $N_D = 0$ $N_D =$ Denominator determinant
8	Slight curved member  Curvature equation for $P = 0$: $u = f \cdot \sin\pi\frac{x}{l}$ $P_E =$ Eulerload $= E J \frac{\pi^2}{l^2}$; $\alpha = \frac{l}{k}$	$P \cdot y = \frac{P \cdot f}{\frac{P_E}{P} - 1} \cdot \sin\pi\frac{x}{l} + A \cos\varphi + B \cdot \sin\varphi$ $P k \cdot y' = \frac{P \cdot f}{\frac{P_E}{P} - 1} \cdot \frac{\pi}{\alpha} \cdot \cos\pi\frac{x}{l} - A \sin\varphi + B \cdot \cos\varphi$ $P k^2 y'' = -\frac{P \cdot f}{1 - \frac{P}{P_E}} \cdot \sin\pi\frac{x}{l} - A \cos\varphi - B \cdot \sin\varphi$	$M_x = P \cdot u + P \cdot y$ $M_x = \frac{P \cdot f}{1 - \frac{P}{P_E}} \cdot \sin\pi\frac{x}{l} + A \cdot \cos\varphi + B \cdot \sin\varphi$ $M_{max} = \frac{P \cdot f}{1 - \frac{P}{P_E}}$ at center	$A = 0$; $B = 0$ $\alpha = 180^0$
9	Slight curved member  Curvature equation für $P = 0$: $u = \xi \cdot (l \cdot x - x^2)$ with $\xi = \frac{4f}{l^2}$	$P \cdot y = A \cos\varphi + B \cdot \sin\varphi - P \cdot \xi (l x - x^2) - 2 P \xi k^2$ $P k y' = -A \cdot \sin\varphi + B \cdot \cos\varphi - P \cdot k \xi (l - 2x)$ $P k^2 y'' = -A \cdot \cos\varphi - B \cdot \sin\varphi + 2 P k^2 \cdot \xi$	$M_x = P \cdot u + P \cdot y$ $M_x = A \cdot \cos\varphi + B \cdot \sin\varphi - 2 P k^2 \cdot \xi$ or $M_x = 2 P \xi \cdot k^2 \left(\cos\varphi + \operatorname{tg}\frac{\alpha}{2} \sin\varphi - 1\right)$ $M_{max} = 2 P \xi k^2 \left(\frac{1}{\cos\frac{\alpha}{2}} - 1\right)$ at center	$A = 2 P \xi k^2$ $B = 2 P \xi k^2 \cdot \operatorname{tg}\frac{\alpha}{2}$ $\alpha = 180^0$

Nr.	Problem graph identification	Deflection $\frac{y}{}$ (P time value) Tangent $\frac{dy}{dx}$ (P·k " ") 2. derivation $\frac{d^2y}{dx^2}$ (P k² " ")	Moments: M_x; $M_x = -EJ\frac{d^2y}{dx^2}$; Special Moment	Constants A and B Member buckles under
10	Slight curved member as 9  Curvature equation for $P=0;\ g=0$ $u=\xi(l\cdot x-x^2)$ mit $\xi=\frac{4f}{l^2}$	$P\cdot y = A\cdot\cos\varphi + B\cdot\sin\varphi - \frac{g}{2}(lx-x^2)$ $-P\cdot\xi(l\cdot x-x^2)-k^2(g+2P\cdot\xi)$ $Pky' = -A\cdot\sin\varphi + B\cdot\cos\varphi$ $-\frac{gk}{2}(l-2x) - P\xi\cdot k(l-2x)$ $Pk^2y'' = -A\cdot\cos\varphi - B\cdot\sin\varphi + k^2(g+2P\cdot\xi)$	$M_{x'} = \frac{g}{2}(l\cdot x - x^2) + P\cdot\xi(lx-x^2) + P\cdot y$ $M_x = A\cdot\cos\varphi + B\cdot\sin\varphi - k^2(g+2P\cdot\xi)$ $M_{max} = k^2(g+2P\cdot\xi)\cdot\left(\frac{1}{\cos\frac{\alpha}{2}}-1\right)$ in center	$A = +k^2(g+2P\cdot\xi)$ $B = k^2(g+2P\xi)\operatorname{tg}\frac{\alpha}{2}$ $\alpha = 180^0$
11	Slight curved member as 9  Curvature equation for $P=0;\ g=0$ $u=\xi\cdot(l\cdot x-x^2);\ \xi=\frac{4f}{l^2}$ Find : M_E	$P\cdot y = A\cdot\cos\varphi + B\cdot\sin\varphi - \frac{g}{2}(lx-x^2)$ $-P\cdot\xi(l\cdot x-x^2)+M_E\left(1-\frac{x}{l}\right)-k^2(g+2P\xi)$ $P\cdot ky' = -A\cdot\sin\varphi + B\cdot\cos\varphi$ $-\frac{gk}{2}(l-2x) - Pk\xi(l-2x) - M_E\frac{k}{l}$ $Pk^2\cdot y'' = -A\cdot\cos\varphi - B\cdot\sin\varphi + k^2(g+2P\cdot\xi)$	$M_x = \frac{g}{2}(l\cdot x-x^2) + P\xi(lx-x^2) - M_E\left(1-\frac{x}{l}\right)+P\cdot y$ $M_x = A\cdot\cos\varphi + B\cdot\sin\varphi - k^2(g+2P\xi)$ $M_E\cdot\left(1-\frac{\alpha}{\operatorname{tg}\alpha}\right) = (gl^2+2P\xi\cdot l^2)\cdot\left(\frac{1}{\alpha}\cdot\operatorname{tg}\frac{\alpha}{2}-\frac{1}{2}\right)$	$A = +k^2(g+2P\xi)-M_E$ $B = k^2(g+2P\xi)\operatorname{tg}\frac{\alpha}{2}$ $+\frac{M_E}{\operatorname{tg}\alpha}$ $\alpha = 257^0\,30'$
12	Slight curved member as 9  Curvature equation for $P=0;\ g=0$ $u=\xi(l\cdot x-x^2);\ \xi=\frac{4f}{l^2}$ Find : M_E	$P\cdot y = A\cdot\cos\varphi + B\cdot\sin\varphi - \frac{g}{2}(l\cdot x-x^2)$ $-P\cdot\xi(l\cdot x-x^2)+M_E-k^2(g+2P\cdot\xi)$ $Pky' = -A\cdot\sin\varphi + B\cdot\cos\varphi$ $-\frac{gk}{2}(l-2x) - Pk\xi(l-2x)$ $Pk^2y'' = -A\cdot\cos\varphi - B\cdot\sin\varphi + k^2(g+2P\xi)$	$M_x = \frac{g}{2}(lx-x^2) + P\xi(lx-x^2) - M_E + P\cdot y$ $M_x = A\cdot\cos\varphi + B\cdot\sin\varphi - k^2(g+2P\xi)$ at center: $M_M = k^2(g+2P\xi)$ $M_E = k^2(g+2P\xi)\cdot\left(1-\frac{\alpha}{2\operatorname{tg}\frac{\alpha}{2}}\right)$ or $\times\left(\frac{\alpha}{2\sin\frac{\alpha}{2}}-1\right)$ $M_E\left(\frac{\alpha}{\sin\alpha}-\frac{\alpha}{\operatorname{tg}\alpha}\right) = (gl^2+2P\xi l^2)\left(\frac{1}{\alpha}\cdot\operatorname{tg}\frac{\alpha}{2}-\frac{1}{2}\right)$	$A = +k^2(g+2P\cdot\xi)-M_E$ $B = [k^2(g+2P\xi)-M_E]$ $\times\operatorname{tg}\frac{\alpha}{2}$ $\alpha = 360^0$
13	 $\varphi=\frac{x}{k};\ \psi=\frac{u}{k};\ \zeta=\frac{b}{k};\ \eta=\frac{d}{k}$ $\alpha=\zeta+\eta=\frac{l}{k};\ k=\sqrt{\frac{EJ}{P}}$	$P\cdot y = A_1\cdot\cos\varphi + B_1\cdot\sin\varphi - Q\frac{d}{l}\cdot x - M_1\left(1-\frac{x}{l}\right) - M_2\frac{x}{l}$ $Pky' = -A_1\cdot\sin\varphi + B_1\cdot\cos\varphi - Q\frac{d}{l}\cdot k + M_1\frac{k}{l} - M_2\frac{k}{l}$ $Pk^2y'' = -A_1\cdot\cos\varphi - B_1\cdot\sin\varphi$ $P\cdot v = A_2\cdot\cos\psi + B_2\cdot\sin\psi - Q\frac{b}{l}\cdot u - M_1\frac{u}{l} - M_2\left(1-\frac{u}{l}\right)$ $Pkv' = -A_2\cdot\sin\psi + B_2\cdot\cos\psi - Q\frac{b}{l}\cdot k - M_1\frac{k}{l} + M_2\frac{k}{l}$ $Pk^2v'' = -A_2\cdot\cos\psi - B_2\cdot\sin\psi$	$M_x = Q\frac{d}{l}\cdot u + M_1\left(1-\frac{x}{l}\right) + M_2\frac{x}{l} + P\cdot y;$ $M_u = Q\frac{b}{l}u + M_1\frac{u}{l} + M_2\left(1-\frac{u}{l}\right) + P\cdot v$ $M_x = A_1\cdot\cos\varphi + B_1\cdot\sin\varphi;$ $M_u = A_2\cdot\cos\psi + B_2\cdot\sin\psi$ with Q the moment is : $M_Q = Q\cdot k\frac{\sin\zeta\cdot\sin\eta}{\sin\alpha} + M_1\frac{\sin\eta}{\sin\alpha} + M_2\frac{\sin\zeta}{\sin\alpha}$ For Q in center and $M_1 = M_2 = M_a$, we get $M_Q = Q\cdot k\frac{1}{2}\operatorname{tg}\frac{\alpha}{2} + M_a\frac{1}{\cos\frac{\alpha}{2}}$	$A_1 = M_1;\ A_2 = M_2$ $B_1 = Q\cdot k\frac{\sin\eta}{\sin\alpha}$ $-\frac{M_1}{\operatorname{tg}\alpha} + \frac{M_2}{\sin\alpha}$ $B_2 = Q\cdot k\frac{\sin\zeta}{\sin\alpha}$ $+\frac{M_1}{\sin\alpha} - \frac{M_2}{\operatorname{tg}\alpha}$ $\alpha = 180^0$

Nr.	Problem graph identification	Deflection y (P time value) Tangent $\frac{dy}{dx}$ ($P\cdot k$ " ") 2. Derivation $\frac{d^2y}{dx^2}$ (Pk^2 " ")	Moments $\begin{cases} M_x \\ M_x = -E\cdot J\,\frac{d^2\cdot y}{dx^2} \\ \text{Special Moments} \end{cases}$	Constants A and B Member buckles under
14	 $\varphi=\frac{x}{k};\psi=\frac{u}{k};\zeta=\frac{b}{k};\eta=\frac{d}{k}$ $\alpha=\zeta+\eta=\frac{l}{k};\ k=\sqrt{\frac{EJ}{P}}$ Find : M_1	$P\cdot y = A_1\cdot\cos\varphi+B_1\cdot\sin\varphi-Q\frac{d}{l}\cdot x+M_1\left(1-\frac{x}{l}\right)$ $Pky'=-A_1\cdot\sin\varphi+B_1\cdot\cos\varphi-Q\frac{d}{l}\cdot k-M_1\frac{k}{l}$ $Pk^2y''=-A_1\cdot\cos\varphi-B_1\cdot\sin\varphi$ $P\cdot v = A_2\cdot\cos\psi+B_2\cdot\sin\psi-Q\frac{b}{l}\cdot u+M_1\frac{u}{l}$ $Pkv'=-A_2\cdot\sin\psi+B_2\cdot\cos\psi-Q\frac{b}{l}\cdot k+M_1\frac{k}{l}$ $Pk^2v''=-A_2\cdot\cos\psi-B_2\cdot\sin\psi$	$M_x=Q\frac{d}{l}x-M_1\left(1-\frac{x}{l}\right)+P\cdot y;\ M_u=Q\frac{b}{l}\cdot u-M_1\frac{u}{l}+P\cdot v$ $M_x=A_1\cdot\cos\varphi+B_1\cdot\sin\varphi;\quad M_u=A_2\cdot\cos\psi+B_2\cdot\sin\psi$ $M_1\left(1-\frac{\alpha}{\operatorname{tg}\alpha}\right)=Q\cdot l\left(\frac{\sin\eta}{\sin\alpha}-\frac{d}{l}\right)$	$A_1=-M_1;\ A_2=0$ $B_1=Q\cdot k\frac{\sin\eta}{\sin\alpha}+\frac{M_1}{\operatorname{tg}\alpha}$ $B_2=Q\cdot k\frac{\sin\zeta}{\sin\alpha}-\frac{M_1}{\sin\alpha}$ $\alpha=257^\circ 30'$
15	 $\varphi=\frac{x}{k};\ \psi=\frac{u}{k};$ $\zeta=\frac{b}{k};\ \eta=\frac{d}{k}$ $\alpha=\zeta+\eta=\frac{l}{k};\ k=\sqrt{\frac{EJ}{P}}$ Find: M_1 nd M_2	y- and v- same as under 13. but M_1 and M_2 of opposite signs.	$M_x=Q\frac{d}{l}x-M_1\left(1-\frac{x}{l}\right)-M_2\frac{x}{l}+P\cdot y$ $M_u=Q\frac{b}{l}\cdot u-M_1\frac{u}{l}-M_2\left(1-\frac{u}{l}\right)+P\cdot v$ $M_x=A_1\cdot\cos\varphi+B_1\cdot\sin\varphi\quad M_u=A_2\cdot\cos\psi+B_2\cdot\sin\psi$ $M_1\left(1-\frac{\alpha}{\operatorname{tg}\alpha}\right)+M_2\left(\frac{\alpha}{\sin\alpha}-1\right)=Q\cdot l\left(\frac{\sin\eta}{\sin\alpha}-\frac{d}{l}\right)$ (1) $M_1\left(\frac{\alpha}{\sin\alpha}-1\right)+M_2\left(1-\frac{\alpha}{\operatorname{tg}\alpha}\right)=Q\cdot l\left(\frac{\sin\zeta}{\sin a}-\frac{b}{l}\right)$ (2) $M_Q=Q\cdot k\,\frac{\sin\zeta\cdot\sin\eta}{\sin\alpha}-M_1\frac{\sin\eta}{\sin\alpha}-M_2\frac{\sin\zeta}{\sin\alpha}$	$A_1=-M_1;\ A_2=-M_2$ $B_1=Q\cdot k\frac{\sin\eta}{\sin\alpha}+\frac{M_1}{\operatorname{tg}\alpha}-\frac{M_2}{\sin\alpha}$ $B_2=Q\cdot k\frac{\sin\zeta}{\sin\alpha}-\frac{M_1}{\sin\alpha}+\frac{M_2}{\operatorname{tg}\alpha}$ $\alpha=360^\circ$
16	 Find : M_1 and M_2; P only for field u/e $y'_1=y'_{(x=0)};\ v'_2=v'_{(u=0)}$ In field $l_1: J_1$ In " $l_2: J_2$ In central field $k=\sqrt{\frac{EJ}{P}};\ \alpha=\frac{l}{k}$	y- and v- same as under 13. but M_1 and M_2 of opposite signs.	$M_x=Q\cdot\frac{d}{l}\cdot x-M_1\left(1-\frac{x}{l}\right)-M_2\frac{x}{l}+P\cdot y;$ $M_u=Q\frac{b}{l}\cdot u-M_1\frac{u}{l}-M_2\left(1-\frac{u}{l}\right)+P\cdot v$ $M_x=A_1\cdot\cos\varphi+B_1\cdot\sin\varphi$ $M_u=A_2\cdot\cos\psi+B_2\cdot\sin\psi$ $M_1\left(1-\frac{\alpha}{\operatorname{tg}\alpha}\right)+M_2\left(\frac{\alpha}{\sin\alpha}-1\right)=Q\cdot l\left(\frac{\sin\eta}{\sin\alpha}-\frac{d}{l}\right)-P\cdot l\cdot y_1'$ (1a) $M_1\left(\frac{\alpha}{\sin\alpha}-1\right)+M_2\left(1-\frac{\alpha}{\operatorname{tg}\alpha}\right)=Q\cdot l\left(\frac{\sin\zeta}{\sin\alpha}-\frac{b}{T}\right)-P\cdot l\cdot y_2'$ (2a) $:\ y_1'=\frac{M_1\cdot l_1}{3\cdot EJ_1};\ y_2'=\frac{M_2\cdot l_2}{3\cdot EJ_2}$ $M_1\left(P\cdot l\cdot\frac{l_1}{3\cdot EJ_1}+1-\frac{\alpha}{\operatorname{tg}\alpha}\right)+M_2\left(\frac{\alpha}{\sin\alpha}-1\right)=Q\cdot l\left(\frac{\sin\eta}{\sin\alpha}-\frac{d}{l}\right)$ (1) $M_1\left(\frac{\alpha}{\sin\alpha}-1\right)+M_2\left(P\cdot l\cdot\frac{l_2}{3\cdot EJ_2}+1-\frac{\alpha}{\operatorname{tg}\alpha}\right)=Q\cdot l\left(\frac{\sin\zeta}{\sin\alpha}-\frac{b}{l}\right)$ (2)	$A_1=-M_1;\ A_2=-M_2$ $B_1=Q\cdot k\frac{\sin\eta}{\sin\alpha}+\frac{M_1}{\operatorname{tg}\alpha}-\frac{M_2}{\sin\alpha}$ $B_2=Q\cdot k\frac{\sin\zeta}{\sin\alpha}-\frac{M_1}{\sin\alpha}+\frac{M_2}{\operatorname{tg}\alpha}$ $N_D=0$ $N_D=$ Denominator determinant

Nr.	Problem graph identification	Deflection y (P time value) Tangent $\frac{dy}{dx}$ ($P\cdot k$ ″ ″) 2. Derivation $\frac{d^2y}{dx^2}$ (Pk^2 ″ ″)	Moments $\begin{cases} M_r \\ M_x=-E\cdot J\,\frac{d^2\cdot y}{d\,x^2} \end{cases}$ Special Moments	Constants A and B Member buckles under
17	 Moments for $P=0$ $\alpha_1=\frac{a_1}{k};\ \alpha_2=\frac{a_2}{k};\ \alpha_3=\frac{a_3}{k}$	$M_1=M_{01}+P\cdot y_1;\quad y_1=\frac{M_1}{P}-\frac{M_{01}}{P}$ $M_2=M_{02}+P\cdot y_2;\quad y_2=\frac{M_2}{P}-\frac{M_{02}}{P}$ $M_1\left(\frac{\nu_1'}{P\cdot a_1}+\frac{\nu_2'}{P\cdot a_2}\right)-M_2\frac{\nu_2''}{P\cdot a_2}-\frac{y_1}{a_1}-\frac{y_1-y_2}{a_2}=0$ $M_1\cdot\frac{\nu_2''}{P\cdot a_2}+M_2\left(\frac{\nu_2'}{P\cdot a_2}+\frac{\nu_3'}{P\cdot a_3}\right)-\frac{y_2}{a_3}-\frac{y_2-y_1}{a_2}=0$ Extended Clapeyron equation from Müller-Breslau, Graphische Statik II Bd., 2. Abteilung 1925, p 643	$M_1\frac{\sin(\alpha_1+\alpha_2)}{\sin\alpha_1\cdot\sin\alpha_2}a_2-M_2\cdot\frac{\alpha_2}{\sin\alpha_2}=M_{01}\left(1+\frac{a_2}{a_1}\right)-M_{02}$ $-M_1\frac{\alpha_2}{\sin\alpha_2}+M_2\frac{\sin(\alpha_2+\alpha_3)}{\sin\alpha_2\cdot\sin\alpha_3}\cdot\alpha_2=-M_{01}+M_{02}\left(1+\frac{a_2}{a_3}\right)$ Solution of equations: $M_1=Q_1\cdot k\frac{\sin\alpha_1\cdot\sin(\alpha_2+\alpha_3)}{\sin(\alpha_1+\alpha_2+\alpha_3)}+Q_2\cdot k\cdot\frac{\sin\alpha_1\cdot\sin\alpha_3}{\sin(\alpha_1+\alpha_2+\alpha_3)}$ $M_2=Q_1\cdot k\frac{\sin\alpha_1\cdot\sin\alpha_3}{\sin(\alpha_1+\alpha_2+\alpha_3)}+Q_2\cdot k\cdot\frac{\sin(\alpha_1+\alpha_2)\sin\alpha_3}{\sin(\alpha_1+\alpha_2+\alpha_3)}$ by analogy (without column effect) $M_{01}=Q_1\frac{(a_2+a_3)\cdot a_1}{l}+Q_2\frac{a_1\cdot a_3}{l}\quad M_{02}=Q_1\frac{a_1\cdot a_3}{l}+Q_2\frac{(a_1+a_2)a_3}{l}$	$\nu'=1-\frac{\alpha}{\operatorname{tg}\alpha}$ $\nu''=\frac{\alpha}{\sin\alpha}-1$
18	 $\alpha_1=\frac{a_1}{k};\ \alpha_2=\frac{a_2}{k};\ \alpha_3=\frac{a_3}{k}$ $\alpha_4=\frac{a_4}{k};\ \alpha=\frac{l}{k}$	Without column effect $M_{01}=A\cdot a_1=Q_1\frac{a_1(a_2+a_3+a_4)}{l}$ $+Q_2\frac{a_1(a_3+a_4)}{l}+Q_3\frac{a_1a_4}{l}$ $M_{02}=A(a_1+a_2)-Q_1\cdot a_2=Q_1\left[\frac{a_1(a_2+a_3+a_4)}{l}\right.$ $\left.+\frac{a_2(a_2+a_3+a_4)}{l}-\frac{a_2(a_1+a_2+a_3+a_4)}{l}\right]$ $+Q_2\frac{(a_1+a_2)(a_3+a_4)}{l}+Q_3\frac{(a_1+a_2)a_4}{l}$ $M_{03}=Q_1\cdot\frac{a_1(a_3+a_4)}{l}+Q_2\frac{(a_1+a_2)(a_3+a_4)}{l}$ $+Q_3\frac{(a_1+a_2)a_4}{l}$ $M_{04}=Q_1\frac{a_1\cdot a_4}{l}+Q_2\frac{(a_1+a_2)\cdot a_4}{l}$ $+Q_3\frac{(a_1+a_2+a_3)a_4}{l}$	Analogous with column effect, $M_1=Q_1\cdot k\frac{\sin\alpha_1\cdot\sin(\alpha_2+\alpha_3+\alpha_4)}{\sin(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$ $+Q_2\cdot k\frac{\sin\alpha_1\cdot\sin(\alpha_3+\alpha_4)}{\sin\alpha}+Q_3\cdot k\frac{\sin\alpha_1\cdot\sin\alpha_4}{\sin\alpha}$ $M_2=Q_1\cdot k\frac{\sin\alpha_1\cdot\sin(\alpha_3+\alpha_4)}{\sin\alpha}$ $+Q_2\cdot k\frac{\sin(\alpha_1+\alpha_2)\cdot\sin(\alpha_3+\alpha_4)}{\sin\alpha}+Q_3\cdot k\frac{\sin(\alpha_1+\alpha_2)\cdot\sin\alpha_4}{\sin\alpha}$ $M_3=Q_1\cdot k\frac{\sin\alpha_1\cdot\sin\alpha_4}{\sin\alpha}$ $+Q_2\cdot k\frac{\sin(\alpha_1+\alpha_2)\sin\alpha_4}{\sin\alpha}+Q_3\cdot k\frac{\sin(\alpha_1+\alpha_2+\alpha_3)\sin\alpha_4}{\sin\alpha}$	
19	 $\varphi=\frac{x}{k};\psi=\frac{u}{k};\zeta=\frac{b}{k};\eta=\frac{d}{k}$ $\alpha=\zeta+\eta=\frac{l}{k}$ $R_1=Q\left(1-\frac{1}{2}\frac{b}{l}\right);R_2=Q\frac{b}{2l}$	1) $P\cdot y=A\cos\varphi+B\cdot\sin\varphi-\mathfrak{M}_x-p\cdot k^2$ $P\cdot k\cdot y'=-A\sin\varphi+B\cos\varphi-Q_x\cdot k$ $Pk^2y''=-A\cos\varphi-B\sin\varphi+p\cdot k^2$ 2) $P\cdot v=C\cos\psi+D\sin\psi-\mathfrak{M}_u$ $P\cdot k\cdot v'=-C\sin\psi+D\cos\psi-Q_u\cdot k$ $P\cdot k^2\cdot v''=-C\cos\psi-D\sin\psi$	$M_x=Q\frac{b}{2}\left(2\frac{x}{b}-\frac{x}{l}-\frac{x^2}{b^2}\right)+P\cdot y;\quad M_u=Q\frac{b}{2}\left(1-\frac{b}{l}\right)\cdot\frac{u}{d}+P\cdot v$ $M_x=\mathfrak{M}_x+P\cdot y\quad M_u=\mathfrak{M}_u+P\cdot v$ In »1« $M_1=p\,k^2\cdot\frac{1-\cos\zeta}{\sin\alpha}\cdot\sin\eta$ $M_x=A\cdot\cos\varphi+B\cdot\sin\varphi-p\,k^2;\quad M_u=D\cdot\sin\psi$	$A=+p\,k^2$ $B=p\,k^2\frac{\cos\eta-\cos\alpha}{\sin\alpha}$ $C=0$ $D=p\,k^2\frac{1-\cos\zeta}{\sin\alpha}$

Nr.	Problem graph identification	Deflection y (P time value) Tangents $\frac{dy}{dx}$ ($P \cdot k$ " ") 2. Derivation $\frac{d^2y}{dx^2}$ (Pk^2 " ")	Moments $\begin{cases} M_x \\ M_x = -E \cdot J \frac{d^2 y}{dx^2} \\ \text{Special Moments} \end{cases}$	Constants A and B Member buckles under
20	 $\zeta = \frac{a}{k};\ \eta = \frac{b}{k}$ $\varphi = \frac{x}{k}$	$P \cdot y = A \cdot \cos\varphi + B \cdot \sin\varphi - Q \cdot x + P \cdot x \cdot y_1'$ $P \cdot k \cdot y' = -A \sin\varphi + B \cos\varphi - Q \cdot k + P \cdot k y_1'$ $P \cdot k^2 \cdot y'' = -A \cos\varphi - B \sin\varphi$ $y_1' = \frac{M_1}{P \cdot b}\left(1 - \frac{\eta}{\operatorname{tg}\eta}\right).$ intermediate values in range ①—② te be treated as under Nr. 1.	$M_{①} = M_0 + Q \cdot a + P \cdot \delta$ $M_{①} = \frac{M_0}{\cos\zeta} + Q \cdot k \cdot \operatorname{tg}\zeta + P \cdot a\, y_1'$ $M_{①} = \frac{M_0/\cos\zeta + Q \cdot K \cdot \operatorname{tg}\zeta}{1 - a/b \cdot \left(1 - \frac{n}{\operatorname{tg}\eta}\right)}$	$A = M_0$ $B = M_0 \cdot \operatorname{tg}\zeta + \frac{Q \cdot k}{\cos\zeta}$
21	 $\varphi = \frac{x}{k};\ \psi = \frac{u}{k};\ \zeta = \frac{b}{k};\ \eta = \frac{d}{k}$ $\alpha = \zeta + \eta = \frac{l}{k};\ k = \sqrt{\frac{EJ}{P}}$ 	Left side: range »b« Right side: range »d« $P \cdot y = B_1 \cdot \sin\varphi - \frac{M}{l} \cdot x$ $P \cdot v = B_2 \cdot \sin\psi - \frac{M}{l} \cdot u$ $Pky' = -B_1 \cos\varphi - \frac{M}{l} \cdot k$ $P \cdot kv' = B_2 \cos\psi - \frac{M}{l} \cdot k$ $Pk^2 y'' = -B_1 \sin\varphi$ $Pk^2 \cdot v'' = B_2 \sin\psi$	Range »b« Range »d« $M_x = \frac{M}{l} \cdot x + P \cdot y$ $M_u = \frac{M}{l} \cdot u + P \cdot v$ $M_x = B_1 \cdot \sin\varphi$ $M_u = B_2 \cdot \sin\psi$ The moment at B is: $M_1 = M \frac{\cos\eta \cdot \sin\zeta}{\sin\alpha}$ $M_2 = \frac{\cos\zeta \cdot \sin\eta}{\sin\alpha}$	$B_1 = M \frac{\cos\eta}{\sin\alpha}$ $B_2 = M \frac{\cos\zeta}{\sin\alpha}$ $\alpha = 180^0$

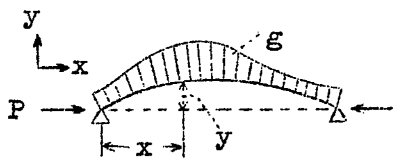


Figure 1.- Beam on two supports under longitudinal load P and transverse forces g.

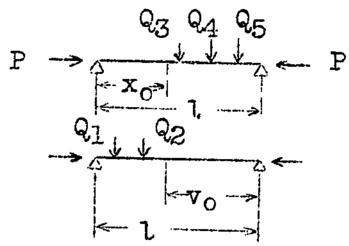


Figure 2.- Beam on two supports under load P and concentrated loads Q_1 - Q_5.

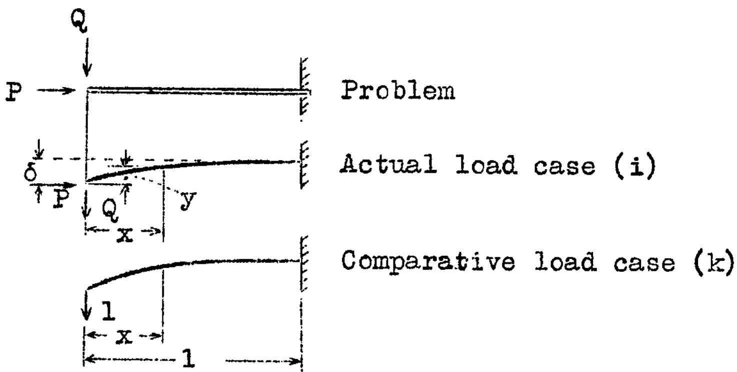


Figure 3.- Built-in beam under load P and Q.